Abstract

This study deals with students' mathematical reasoning and their approaches to proof and proving. Research on proof and proving is considered critical in mathematics and mathematics education. In particular, flaws in understanding the interplay between empirical and formal aspects of mathematics are regarded as one of the main sources of students' difficulties with proving. Empirical aspects of mathematics rely heavily on examples, which are considered useful tools in mathematical inquiry. Yet, examples play different roles in proving.

The purpose of the study was to examine and characterize students' understanding of the interplay between examples and statements as well as the roles of mathematical examples and counterexamples in proving, i.e., their understanding of the status of examples in determining the validity of mathematical statements. This kind of understanding entails making a distinction between universal and existential statements, and realizing that although a single example (or even a large collection of examples) is not sufficient for proving a universal statement. At the same time, a single example is sufficient for proving an existential statement. Moreover, a single counterexample is sufficient for refuting a universal statement but it is not applicable for refuting an existential statement. This kind of understanding is critical for proving. However, it is not explicitly addressed in the school curriculum and is left to students to develop indirectly mostly on their own. The nature of this understanding has not been fully conceptualized, nor has it been studied systematically prior to this study.

For the purpose of the study, a conceptual framework describing the interplay between examples and counterexamples in proving and disproving was developed. This framework presents the inter-connections between four types of examples (confirming, non-confirming, contradicting, and irrelevant) and the validity of two types of mathematical statements (universal and existential). "Understanding of the status of examples in determining the validity of mathematical statements" was defined in this research as becoming fluent with the conceptual framework, that is, with the logical inferences that can and cannot be made based on the different types of examples with respect to the validity of the two kinds of statements, according to the conceptual framework.

This framework provided the basis for designing six special types of mathematical tasks that both assess students' understanding and at the same time facilitate the development of their

understanding. Each task addressed at least one aspect of the roles of examples in proving, according to the conceptual framework. The tasks were constructed in two parallel versions (in algebra and geometry) and drew on the mathematics secondary school curriculum. The tasks aimed at evoking uncertainty and doubt regarding a truth-value of mathematical statements, and consequently – the need to prove or disprove.

The tasks were implemented with six pairs of top-level high-school students from two distinct schools in the northern area of Israel. Each pair of students participated in a series of six semi-structural task-based interviews, during which students coped with the different mathematical tasks.

Data collection included video recordings of the interviews (about 60 hours in total), students' written work and field notes. The data were analyzed using qualitative research methodology, which resonated with the goals of the study. Based on the conceptual framework, students' written work and utterances were analyzed in terms of the manifestations of their understanding (or misunderstanding) of the status of examples in determining the validity of mathematical statements.

The findings provide a complex account of students' understanding of the roles of examples in proving and reveal various inconsistencies in their understandings. The main findings include:

1. Students' understanding of the role of examples in determining the validity of mathematical statements is closely connected to their understanding of the mathematical statement at hand.

   With respect to students' understanding of mathematical statements, three major difficulties were identified: (i) a difficulty to detect the parts of the statement; (ii) a difficulty to determine the logical status of the parts of the statements; (iii) a difficulty to detect the type of statement, which results in a tendency to treat an existential statement as universal.
2. Students' understanding of the role of examples in determining the validity of mathematical statements is closely connected to their understanding of the concepts associated with the types of examples.
3. Inconsistencies in students' understanding of the role of examples in determining the validity of mathematical statements were detected. Some were manifested as discrepancies between the views students explicitly expressed and their actual

performance. Others were manifested as discrepancies between students' performances on similar types of tasks.

4. Many students exhibited fallacious reasoning that was based on sophisticated mathematical ideas that were incorrectly or inaccurately applied. These include: (i) relying on systematic examination of examples to support their inference; (ii) relying on arbitrariness of choice of example to support their inference; (iii) relying on a "distorted" theorem to support an assertion regarding the validity of a statement; (iv) inferring from their failure to find a certain type of example that such example does not exist.
5. Engaging in the different kinds of tasks elicited students' understanding of logical connections between examples, statements and proving.

The theoretical significance of the study rests mainly on the characterization of students' understanding of the roles of examples in proving and the identification of the inconsistencies in their understanding. The conceptual framework that was developed within this study proved to be a useful tool that can be used to assess students' understanding of the logical connections between examples and statements and to construct mathematical tasks that can support and promote the development of this understanding.

The research instruments provide six prototypes of tasks that elicit the logical connections between examples, statements and proving. The prototypes can be adapted to different mathematical topics and for various populations of students.