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ORIGINAL ARTICLE

Is this a coincidence? The role of examples in fostering a need for proof

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Abstract It is widely known that students often treat examples that satisfy a certain universal statement as sufficient for showing that the statement is true without recognizing the conventional need for a general proof. Our study focuses on special cases in which examples satisfy certain universal statements, either true or false in a special type of mathematical task, which we term "Is this a coincidence?". In each task of this type, a geometrical example was chosen carefully in a way that appears to illustrate a more general and potentially surprising phenomenon, which can be seen as a conjecture. In this paper, we articulate some design principles underlying the choice of examples for this type of task, and examine how such tasks may trigger a need for proof. Our findings point to two different kinds of ways of dealing with the task. One is characterized by a doubtful disposition regarding the generality of the observed phenomenon. The other kind of response was overconfidence in the conjecture even when it was false. In both cases, a need for "proof" was evoked; however, this need did not necessarily lead to a valid proof. We used this type of task with two different groups: capable high school students and experienced secondary mathematics teachers. The findings were similar in both groups.

1 Introduction

The importance of proof in mathematics education as a reflection of the centrality of proof in mathematics has been widely acknowledged by the mathematics education community. At the same time, students and teachers encounter various difficulties with understanding and construction of mathematical proof (e.g., Healy & Hoyles, 2000; Mariotti, 2006). Some of these difficulties are rooted in students' lack of understanding of the purpose of proof (Balacheff, 1990). In Harel's (2007) terms, the necessity principle is often violated with respect to proof and proving. One possible way to create instructional situations in which an intellectual need for proof arises intrinsically is to use tasks that evoke uncertainty and doubt (Zaslavsky, 2005). Yet, designing such tasks in the context of secondary school mathematics is a non-trivial task for researchers and mathematics educators. Zaslavsky (2008) asserts that for teachers to gain appreciation of the potential of uncertainty in raising the need to prove, they should engage in tasks that evoke uncertainty for them as learners and problem solvers. Thus, the construction of such tasks presents even a greater challenge for the designer.

In this article, we present and discuss one type of task that has a potential to evoke uncertainty and doubt and to promote an intellectual need for proving as a way to resolve it. We term this type of task "Is this a coincidence?". This type of task was originally designed in order to study how high school students understand the role of examples in proving. For each task of this type, an example was carefully chosen in a way that appears to illustrate a more general and potentially surprising phenomenon. The formulation of such phenomenon as a general one can be seen as a conjecture. The uncertainty is evoked by asking whether the observed mathematical phenomenon occurred 'by accident' (i.e., the example is just a special case) or whether it is not a coincidence (i.e., the example is generic and represents a general case satisfying a general rule or property).

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Students' tendency to treat examples that confirm a mathematical claim as sufficient for proving is well documented in the mathematics educational literature (e.g., Fischbein & Kedem, 1982; Harel & Sowder, 2007; Buchbinder & Zaslavsky, 2007). Our experiences with implementing "Is this a coincidence?" type of task indicate that sometimes students' reaction to confirming examples may be quite different; depending on the individual's confidence regarding the truth of a mathematical claim, confirming examples may evoke uncertainty and create a need to resolve it by finding a proof.

2 Methodology

The aim of this study is to articulate some design principles of "Is this a coincidence?" type of task, and to document its contribution in terms of evoking both students' and teachers' appreciation of and need for proving.

For this purpose, we designed multiple versions of "Is this a coincidence?" type of task. All the tasks had the same structure but varied in their mathematical content.[1] We examined the tasks with various populations of students, pre-service and in-service teachers.

In this paper, we present findings that were obtained on two different occasions in which the tasks were implemented: one with students and one with teachers.

2.1 Research instrument: the task "Is this a coincidence?"

A task that we term "Is this a coincidence?" presents a hypothetical student's actions and observation based on his/her experience with a single (geometrical) example. The main question to be addressed in the task is whether the observed phenomenon is a coincidence or not, i.e., whether it holds for every relevant case or just for some specific cases, one of which the student examined. In order to complete the task successfully, one would need either to prove that the described phenomenon is a general one, or to construct a counterexample showing that there are other relevant cases for which the observed phenomenon does not hold. Note that there is no explicit requirement in the task to prove any claim. The intention was to draw the participants' attention to the extent of generality of the phenomenon, and to study the ways in which they deal with the task, focusing on their need to form an assertion and justify it by means of proof or refutation.

Tables 1 and 2 provide examples of the tasks that were developed and used in our study.

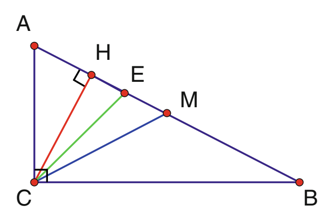
[1] We also created and examined algebraic versions of this type of task, but we do not discuss them here.

We avoided the issue of non-existent cases (Zodik & Zaslavsky, 2008); thus, all the tasks present *possible* phenomena, some of which are necessary outcomes of the described situation (e.g., Task 1 in Table 1, Task 2 in Table 2), while the others (Task 2 in Table 1, Tasks 1 and 3 in Table 2) are not necessary and hold just for some cases. We have termed the latter "a coincidence".

The tasks are based on the secondary school geometry curriculum (in Israel), in order to assure that students and teachers have the sufficient mathematical knowledge for dealing with the tasks. However, instead of stating a general fact and asking to prove it (as most commonly used tasks require), the tasks present situations in which the extent of the generality of the observed phenomena is yet to be determined. The phenomenon itself and the framing of the task were unfamiliar to the participants. Thus, the tasks had a potential of generating surprise and creating uncertainty with respect to the scope of the phenomena for both students and teachers. The alternation of the tasks (coincidence, non-coincidence) was intended to elicit and reinforce the sense of uncertainty.

Table 1 Two (out of four) tasks that were used with students

1. The right triangle task

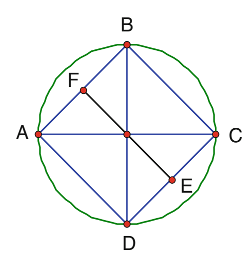


A student performed the following actions:

In a non-isosceles right triangle ΔABC (∢C = 90°) he drew the median to the hypotenuse (CM), the height to the hypotenuse (CH) and the right angle bisector (CE). He noticed that CE lies between CH and CM. He also measured the angles of the triangle ΔHCM and found out that CE bisects ∢HCM.

Is this a coincidence?

2. The inscribed quadrilateral task



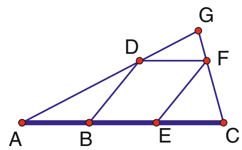
A student performed the following actions:

She constructed a quadrilateral ABCD with perpendicular diagonals, inscribed in a circle. From F, the midpoint of AB, she drew a segment that passed through the point of intersection of the diagonals and intersects the opposite side of the quadrilateral—CD at point E. The student measured segments and found out that E is the midpoint of DC.

Is this a coincidence?

Table 2 Three (out of six) tasks that were used with teachers

1. The perpendicular rays task

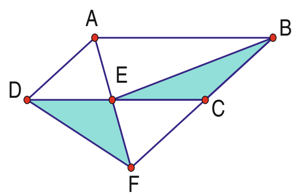


A student performed the following actions:

He divided a segment AC into three equal parts: AB, BE, EC. Then he constructed a parallelogram BDFE. He drew rays AD and CF and marked their intersection point G. He measured ∢G and found out that these rays are perpendicular.

Is this a coincidence?

2. The triangle areas task

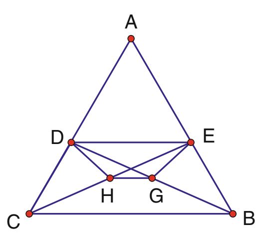


A student performed the following actions:

On the side CD of a parallelogram ABCD she chose an arbitrary point E. She marked the intersection of AE and BC by F. Then she calculated the areas of the triangles ΔBCE and ΔDEF and found out that the areas are equal.

Is this a coincidence?

3. The isosceles triangle task



A student performed the following actions:

In an isosceles triangle ΔABC (AB = AC) he drew two non-perpendicular segments of equal length: BD and CE (with D on AC and E on AB). The student marked the mid-points of these segments as G and H, respectively. He examined the quadrilateral DEGH and found out that it is a trapezoid.

Is this a coincidence?

2.2 Data collection

The data for this article come from two sources, i.e., two groups of participants. The first group consisted of six pairs of top-level 10th grade students from two Israeli schools. By "top-level" students, we mean students who studied mathematics according to the most extended curriculum offered in senior-high schools in Israel. The students were selected based on their willingness to participate in the study and their articulateness. Within the top-level population, they may be considered rather typical. The reason for selecting top-level students is that difficulties encountered by such "rich but not extreme" cases could imply that similar findings may occur in other populations as well (Patton, 2002).

The data were collected through task-based semi-structured interviews with each pair of students. Students worked cooperatively on the task during a session that lasted about 1.5 h and solved such tasks, two of which appear in Table 1. All sessions were videotaped and transcribed. Students' written work provided an additional source of data.

The second group consisted of six experienced and mathematically knowledgeable secondary teachers, from different schools. Data were collected in a 2-h workshop in which they participated. The teachers received six tasks, three of which appear in Table 2, and worked in pairs or individually prior to a whole group discussion. The role of the facilitator was to monitor and manage the discussion, summarize different points of view and pose questions. The workshop was videotaped.

In addition to a printed version of the tasks that was used in the workshop, in several occasions, the facilitator used Dynamic Geometry (DG) demonstrations (constructed in the Geometer's Sketchpad environment). These demonstrations presented the dynamic figure related to the task, showing the measures of the elements, which are relevant to the task; for instance, the areas of the triangles ΔDEF and ΔBEC in Task 2 in Table 2. The aim of the demonstrations was to provide additional empirical information on the phenomenon in the task. They were shown to the teachers in two cases: when teachers appeared stuck and were not able to proceed with their investigation without additional evidence; or when teachers completed the task successfully, yet exposure to additional and different examples of the phenomenon could stimulate further discussion.

2.3 Data analysis

The data were analyzed in the spirit of the grounded theory (Strauss and Corbin 1998). The categories emerged through a cyclical process of looking into the data, interpretation, re-organization and theme-seeking. Though data were collected through variety of scourers and methods, the analysis and the development of the categories were conducted through the same techniques and focused on two main criteria: (1) utterances that indicated uncertainty encounters and (2) evidence for an evoked need for proving.

We describe the categories here and illustrate them further in the next section.

Table 3 Categorization of the six cases that are presented in this paper

Strength of confidence Participants	Strong uncertainty	Moderate uncertainty	Strong confidence
Students	Case 1:	Case 2:	Case 3:
	Correct proof	Correct disproof	Incorrect "proof"
	The right triangle task (Task 1, Table 1)	The inscribed quadrilateral task (Task 2, Table 1)	The inscribed quadrilateral task (Task 2, Table 1)
Teachers	Case 4:	Case 5:	Case 6:
	Correct proof	Correct disproof	Incorrect "proof"
	The triangle areas task (Task 2, Table 2)	The perpendicular rays task (Task 1, Table 2)	The isosceles triangle task (Task 3, Table 2)

We organize the findings regarding students' and teachers' solutions according to the degree of confidence (or uncertainty) that was evoked by the tasks (similar to Table 3). The categories *strong confidence* and *strong uncertainty* present the two endpoints of the continuum. *Strong confidence* is characterized by explicit comments regarding the generality of the observed phenomenon, e.g., "I'm sure it is a coincidence" or "it has to be so". In case of strong confidence, the decision 'coincidence' or 'not a coincidence' was quickly obtained and accepted as correct (almost) without questioning.

On the other end of the continuum, there is a category *strong uncertainty*. As described above, the design of the tasks involved the use of mathematical phenomena, which were unfamiliar and even surprising for the participants. Thus, in cases when there was no immediate answer whether the observed phenomenon is a coincidence or not, at least some degree of uncertainty was evoked. What made this uncertainty *strong* was the large number of indicators of uncertainty encounters, such as the amount of time spent on the task, the number of unsuccessful attempts to reach a conclusion of any kind, and the number of switches between attempts to prove or to refute the general conjecture suggested by the task. In accordance with Hadass and Hershkowitz (2002), we regard such behaviors as indicators of uncertainty. In addition, these behaviors were accompanied by the participants' explicit expressions of their uncertainty regarding the task, such as "how is it possible?!", "it makes no sense!", "I just can't see it".

Between the two ends of the uncertainty-confidence continuum, there are different cases in which an evoked uncertainty was detected but it could not be characterized as strong. These are cases in which the participants did not come up with any initial assertion regarding the task. On the contrary, they were contemplating whether the observed phenomenon represents a general rule or not. At the same time, the participants did not get stuck or lost during their exploration and were eventually able to resolve their uncertainty without any external intervention. Such cases were categorized as *moderate* uncertainty.

In this article, we present four cases in which the participants created a proof (or disproof) of the conjecture, suggested by the task, as means to resolve the evoked uncertainty.

We also include two cases in which instead of the intended uncertainty the participants experienced a strong sense of confidence, and even overconfidence that the observed phenomenon is *not* a coincidence. In other words, they were sure that the general conjecture, suggested by the task, is true. In these two cases, a strong confidence in the correctness of the conjecture at hand and an urge to prove it resulted in creation of an incorrect "proof"—a flawed argument that aimed at showing that the false conjecture is true. We discuss two such cases: one that was experienced by students and one by teachers.

Altogether, we present in detail six cases that provide insights into ways in which students and teachers deal with the above tasks, with respect to the goals of our study.

We start with findings regarding students' interactions with the tasks that appear in Table 1 and continue with findings regarding teachers' solutions to the tasks in Table 2. Table 3 offers a structure and overview of the different cases.

The collection of the cases that we present in this paper illustrates the variation within the data, and points to some similarities between students and teachers.

Note that the participants' reactions to the tasks (as appear in Table 3) as well as their solutions presented in this paper may not 'cover' all possible cases. Other cases are possible. For example, moderate uncertainty may result in correct proof, or strong uncertainty might lead to incorrect solution—proof or disproof as well. We focus on the cases that occurred in our study.

3 Findings

All 12 students who participated in our study solved correctly the right triangle task (Task 1, Table 1), but only two pairs of students detected that the inscribed quadrilateral task (Task 2, Table 1) presents a phenomenon, which is a coincidence. One pair of students could not resolve the uncertainty evoked by this task and did not complete it; and another three pairs of students provided incorrect answers.

Teachers contemplated a lot on some of the tasks but eventually solved all of them correctly, except for the isosceles triangle task (Task 3, Table 2). In this task, they falsely assumed and incorrectly justified that the quadrilateral DEGH has to be a trapezoid.

We turn now to the presentation of the specific cases.

3.1 Case 1: students' strong uncertainty leading to correct proof (the right triangle task)

The phenomenon observed in this task is not a coincidence. It can be shown by simple calculation of angle measures that the segment CE bisects the angle ∢HCM. One way to start the solution is by articulating the special property of the median to the hypotenuse of a right triangle (CM = AM = MB), following that ΔCMA is an isosceles triangle. From here, the solution is rather straightforward.

However, it was not obvious for the students how to choose a starting point or how to proceed from there, as appears in the case of Kerry and Ben.

When Kerry and Ben started working on this problem, they stated that they were not sure whether the observed phenomenon is a coincidence or not. Initially, Kerry and Ben constructed their own triangle following the description in the task. Ben suggested assigning specific values for the measures of angles A or B: "not something too restricting like 30° or 60° but something arbitrary, like 37.5°". He then assigned values 35° and 55° for the angles A and B, respectively, and started to calculate other angles in the drawing, but got confused and could not complete the solution.

Kerry suggested measuring angles ∢HCE and ∢MCE with a protractor. Ben rejected her suggestion by replying: "But this doesn't give us anything in general. It's still a specific example". Kerry and Ben also tried to substitute α for ∢B, and express other angles with α. At some point, they even wrote that it leads to: ∢HCE $= 45° - \alpha$ and ∢MCE $= 45° - \alpha$, but for some reason they did not notice that this actually proves the conjecture that CE bisects ∢HCM (Fig. 1).

Ben explained: "We still have α here. I don't know why. If we would get like $\alpha = 135°$, it would mean that there is only one such case and the result is a coincidence. But if α would get canceled in the equation, it would mean that this holds for any α. But α is still there… for now."

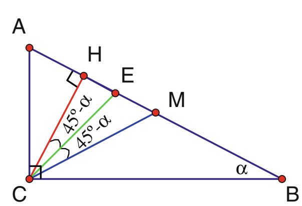


Fig. 1 A possible solution to the right triangle task

Ben and Kerry spent about 30 min contemplating whether this was a coincidence or not. They moved back and forth between trying specific angle values and using general expressions. During all this time, they expressed their uncertainty regarding whether the phenomenon is accidental or not by saying alternately: "this doesn't make sense" and "it seems so likely". Finally, Ben and Kerry succeeded to show that in the right angle triangle with angles 30° and 60°, the angles ∢HCE and ∢MCE are both equal to 15°. Using this case as a generic one, the students were able to refer to their former solution ∢HCE = ∢MCE $= 45° - \alpha$ and recognize it as the main step towards proving that CE bisects ∢HCM. When Ben and Kerry finally showed that the phenomenon is 'not a coincidence' they were very excited and asked each other: "How could we not have seen it earlier?".

In this case, the task and the example provided in it evoked both a strong sense of uncertainty in the presented phenomenon and a strong need to resolve it by proving or refuting the conjecture. In order to resolve the uncertainty, students applied both empirical and general arguments. On the one hand, they constructed several examples using specific numerical values. On the other hand, they were aware of limitations of specific examples for justifying that a certain phenomenon is not a coincidence. That explains why Kerry and Ben used general considerations until the complete proof was obtained.

3.2 Case 2: students' moderate uncertainty leading to correct disproof (the inscribed quadrilateral task)

The specific example in this task is a special case of a quadrilateral with perpendicular diagonals, inscribed in a circle. Such quadrilateral does not necessarily have a diameter as its diagonal; alternatively, it could have one (if it is a kite), and it could have two (if it is a square, which is a special case of a kite). Thus, the observed phenomenon does not hold for any construction described in the task, only for a very special case, i.e., a square. It follows that the observation is a coincidence.

Figure 2b can be seen as a counterexample (this construction in a DGE can be rather compelling). It shows that for a generic kite, the segment connecting the midpoint F of one side (AB) with the intersection of the diagonals does not necessarily intersect the opposite side (DC) at its midpoint. For this to occur, the quadrilateral must be a square (Fig. 2a). It is possible to show that if the quadrilateral is a square then E must be the midpoint of DC. It is also possible to show the converse holds; i.e., if E is in fact the midpoint of DC, then the quadrilateral ABCD must be a square.

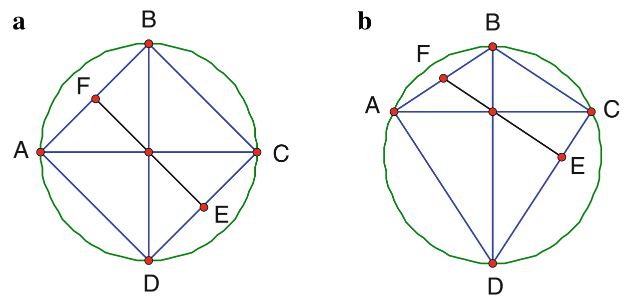


Fig. 2 **a** A special case of a quadrilateral with perpendicular diagonals, inscribed in a circle ("a coincidence"). **b** A more general case of a quadrilateral with perpendicular diagonals, inscribed in a circle ("a counterexample")

The example provided in the task (Fig. 2a) appeared to be very compelling for some students. Others, on the other hand, were more cautious and expressed their uncertainty by making remarks that the observed phenomenon might be accidental and specific to a square. This was the case of Alan and David.

Alan and David started their exploration by highlighting the features of the quadrilateral that they considered critical for the problem: having perpendicular diagonals and being inscribed in a circle. They decided to construct several quadrilaterals, which are not squares, and to check if they have all the required features. Figure 3a and b presents their original work.

First, Alan constructed a (generic, non-square) rectangle inscribed in a circle, but David pointed out that the rectangle does not satisfy the conditions of the problem since its diagonals are not perpendicular. Thus, they rejected it as irrelevant to the task. The second quadrilateral they checked was a rhombus. It had perpendicular diagonals, but in order to be inscribed in a circle, it also had to be a square, so the rhombus was abandoned too.

At this point, David suggested checking a kite, despite Alan's objections that "it's impossible to inscribe a kite in a circle". David started by drawing a circle with a diameter and a chord perpendicular to it. By connecting the endpoints of the diameter and the chord, he managed to construct a kite inscribed in a circle. This was enough for Alan to infer that the phenomenon presented in the task is a coincidence. Alan added to David's drawing a segment (Fig. 3b), which starts at a midpoint of one of the sides and passes through the intersection of the diagonals, and noticed that it did not meet the opposite side of the kite at its midpoint.

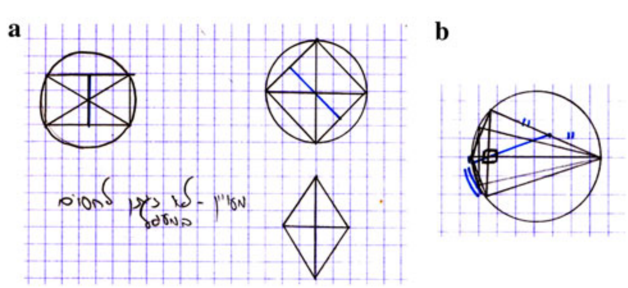


Fig. 3 **a** Different quadrilaterals that Alan and David constructed for the inscribed quadrilateral task. They wrote: "a rhombus—can't be inscribed in a circle". **b** A counterexample constructed by Alan and David

Alan and David actually constructed a counterexample (Fig. 3b) and by this refuted the conjecture implicitly suggested by the task. Note that Fig. 3b shows two kites. After David constructed the first (the larger one), Alan addressed the generality of this counterexample by drawing an additional (smaller) kite.

Interestingly, only the students, who like Alan and David constructed their own examples based on the given properties, were able to complete the task successfully. Those who relied solely on the example of a square provided in the task were not able to solve the task correctly.

Alan and David's approach to the solution indicates that they experienced some sense of uncertainty, which can be viewed as moderate. Initially, they did not seem to have any sense whether the phenomenon is accidental or not. At the same time, they were suspicious and did not want to rely on the specific example that looked like a square. They referred to it as "too perfect" and that "anything can work out for a square". The uncertainty evoked by the task led them to search for the solution and eventually resulted in the construction of the counterexample.

However, this was not the case for Gila and Tami, who dealt with the same problem in a completely different manner, described in the next section.

3.3 Case 3: students' strong confidence leading to incorrect "proof" (the inscribed quadrilateral task)

For Gila and Tami, the task evoked a different reaction than for Alan and David. Consider the following dialog, which took place just a few seconds after Gila and Tami read the task.

Gila: This is not a coincidence.
Tami: How do you know that?
Gila: It's always like this. OK…if this passes through the intersection of the diagonals, it must bisect this angle of the rhombus, the 90°.
Tami: But we don't know that it's a rhombus. It's just some quadrilateral.
Gila: But it says "perpendicular diagonals", and a quadrilateral with perpendicular diagonals is what? A rhombus! I'm not just saying that. So, the diagonals also bisect the angles, we get here isosceles triangles and this segment [points to FE]

is a median to DC. E is a midpoint. That's it. We've proved it.

Tami: OK. Let's write it down. ABCD is a rhombus.

Int.: Can you explain this again?

Tami: It's by definition.

Int.: What do you mean?

Tami: There is a theorem… a quadrilateral with perpendicular diagonals is a rhombus. And it has a proof, so we don't have to prove it again.

Gila and Tami started writing down their justification, but soon discovered that they could not complete it without showing that the supposed rhombus ABCD has at least one right angle. After a short discussion, the girls decided that: "If we want a rhombus to be inscribed in a circle it has to be a square. You can't draw it unless it's a square!". Thus, Gila and Tami inferred that the quadrilateral ABCD is necessarily a square and said that they were certain about that. Based on that, the girls produced a complete written proof that the segment EF connects the midpoints of the opposite sides of the supposed square ABCD.

We would like to note that Gila and Tami are both top-level students. Moreover, both had exhibited sound knowledge of quadrilaterals. For example, in an earlier interview, they were asked to determine the validity of the following statement: "A quadrilateral with diagonals that are of equal length and perpendicular to each other is a kite". They both successfully refuted this false statement by constructing a counterexample. Thus, Gila and Tami had recent (and quite successful) experience with quadrilaterals that have perpendicular diagonals. Surprisingly, they were unable to build on their mathematical knowledge and prior experiences in solving the current task.

In solving other geometry problems, Gila had often mentioned that in geometry it is very important not to rely on visual information. This stands in sharp contradiction to their approach to this task. We attribute this to a strong sense of confidence that the girls had about this task, namely, that the phenomenon is not a coincidence. This confidence, which was probably evoked by the specific example that appeared in the task, led them to confuse a correct and well known theorem that the diagonals of a rhombus are perpendicular, with an incorrect inverse claim that any quadrilateral with perpendicular diagonals is a rhombus. As a result, an incorrect argument showing that the observed phenomenon is not a coincidence was formulated and accepted by them as proof.

3.4 Case 4: teachers' strong uncertainty leading to correct proof (the triangle areas task)

In this task, a hypothetical student begins with a parallelogram and constructs two triangles ΔDEF and ΔBEC related to it, which, she claims, have the same area. The construction seemed arbitrary to the teachers, who appeared to have no initial feeling whether the result is accidental or not, and no intuition to build on. Thus, they were uncertain regarding whether this is a coincidence or not.

The teachers' initial approach was to try to prove that any such triangles have equal areas. For this, they expressed a relationship that would hold if triangles ΔDEF and ΔBEC indeed had equal areas, and one teacher (Stella) wrote this on the board, using symbols to denote lengths of segments (Fig. 4).

The teachers did not know how to proceed further, since they could not think of any explanation why this property holds. This reinforced their initial feeling of uncertainty.

The following excerpt illustrates their growing perplexity.

Stella: [referring to what she wrote on the board, as appears in Fig. 4]…Is it always true? I don't think so. Let's say, I don't have a feeling that it's always true.

Debora: I started to think that it's not true at all. … I couldn't find anything certain.

Facilitator: Does it mean that you couldn't show that it's false?

Debora: No [I couldn't].

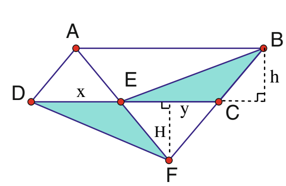
Natalie: The real question is why should it be true?

Dafna: I don't see any reason for that!

Note that even though it was explicitly stated in the task that the student found by measuring that the triangles have equal areas, teaches began doubting even the possibility of that, since they could not either confirm or refute this claim (see Debora's remark).

This was a moment of impasse for the teachers. In order to proceed further the facilitator used a DG demonstration. It presented some dynamic figures related to the task, showing for each one the measures of areas of the triangles ΔDEF and ΔBEC. The facilitator used the dragging

Fig. 4 Teachers' initial approach to the triangle areas task



If triangles ΔDEF and ΔBEC had equal areas, then:

$$yh = xH$$

$$or: \frac{DE}{EC} = \frac{h}{H}$$

function of DG environment to show that the triangles always have the same areas. The next excerpt illustrates the level of surprise that this demonstration evoked.

Facilitator: Here. This measures the areas. Now I'll drag point E….
Natalie: Wow!
Debora: It's true! It's true!
Ronit: We should applaud.
Facilitator: Can you see the values (of areas)?
Natalie: It doesn't matter! The areas are equal! … This means that the equation we've obtained earlier [in Fig. 4] must hold…. But why?

The DG demonstration helped remove the uncertainty associated with the possibility that the triangles could have the same area, as Natalie stated later: "The computer convinced me". In addition, it provided an empirical basis to conjecture that all triangles constructed in such a way have equal areas. However, the uncertainty towards the reasons for that remained, and evoked new enthusiastic attempts to come up with a proof that would provide an explanation why this phenomenon is imperative.

This appeared to be a non-trivial task for teachers. They kept expressing their astonishment after each unsuccessful proving attempt by such utterances: "Amazing!"; "It seemed so unlikely before [DG demonstration]"; "I tried to move point E to the other side, but I still can't see it", and their urge to prove it: "It's so intriguing!", "Why is this property imperative? … Apparently all the triangles should have it!"; "Why should it be so?"

Finally, Dafna suggested a way to prove the area equivalence of the triangles and presented her idea of the proof to the whole group.

> The area of triangle ΔADF equals half of the area of the parallelogram ABCD, since the triangle's base is the side AD, and its height is the distance between two parallel sides AD and BC. By similar arguments, the sum of the areas of triangles ΔAED and ΔBEC is also equal to half of the area of ABCD. Thus the areas of the triangles ΔBEC and ΔDEF are both equal to half of the area of the ABCD minus the area of the triangle ΔAED [Fig. 4].

The teachers were very impressed by Dafna's proof, which provided the explanation why the observed phenomenon holds for any two triangles that are constructed in the way described in the task.

3.5 Case 5: teachers' moderate uncertainty leading to correct disproof (the perpendicular rays task)

In this task, the phenomenon observed by the hypothetical student is accidental, as it does not hold for any parallelogram. It can be shown that the rays AG and CG are perpendicular only if quadrilateral BDFE is a rhombus.

At first, the teachers did not express confidence in their answer. They worked quietly for several minutes trying to establish the extent of generality of the phenomenon. Rather soon they found the "missing property" (i.e., that for the phenomena to hold, the following condition is necessary: BE = BD = EF = FD). This can be characterized as moderate uncertainty regarding the task.

Another aspect of the task that evoked teachers' uncertainty had to do with some logical subtleties surrounding the realization of the solution.

Dafna pointed out that the fact that they could not complete the proof that AG and CG are perpendicular without assuming that BDFE was a rhombus does not imply that the rays are perpendicular *only* if BDFE is a rhombus. "This might be just another step in the proof" she said. Instead, she suggested reproducing the given construction "without the 'missing' property", specifically requiring that the sides of the parallelogram BDFE are of unequal length (DB ≠ DF). In case of 'success', she maintained, it would show that the construction is possible without rays AG and CG being perpendicular.

Based on this suggestion, teachers constructed a general case that constituted a counterexample showing that it is not imperative for AG and CG to be perpendicular (Fig. 5).

In spite of reaching an agreement on a response, the teachers were very engaged in finding a convincing way to prove that the property observed in the task is a coincidence. Their uncertainty regarding this aspect was the one that fostered their attempts to disprove the conjecture by constructing a general counterexample. In Peled and Zaslavsky's (1997) terms, a 'general' counterexample is an example that suggests a way to generate additional counterexamples. Debora even pointed out that "extreme cases", in which the difference in the lengths of adjacent sides of the parallelogram (DB and DF) is apparent, can be very convincing in showing that the rays AG and CG are not necessarily perpendicular.

Only after constructing a general counterexample, the teachers were satisfied with it as a sufficient justification that the phenomenon is indeed a coincidence.

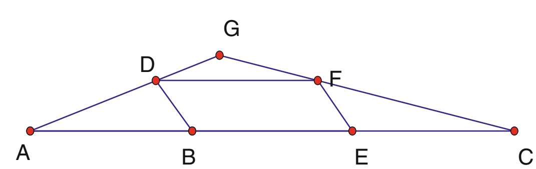


Fig. 5 A general counterexample for the perpendicular rays task

3.6 Case 6: teachers' strong confidence leading to incorrect "proof" (the isosceles triangle task)

In this task, the phenomenon observed is accidental.

One way to perform the construction described in the task is by drawing two arcs of the same radius from the base vertices B and C of the isosceles triangle ΔABC, with points D and E on the sides of the triangle AC and AB, respectively. Thus, two cases are possible: each arc either intersects the side of the triangle at one point (similar to Fig. 6a) or at two points (as in Fig. 6b).

In cases such as Fig. 6a, the two equal length segments CE and BD are symmetrical (with respect to the base height of the triangle). However, in cases such as Fig. 6b, there is a freedom of choice: For example, CE_1 and BD constitute the same case as Fig. 6a, while CE and BD also satisfy the given in the task, but are not symmetrical.

In cases like Fig. 6a, due to the symmetrical properties of the isosceles triangle, the resulting quadrilateral DEGH is a trapezoid (Fig. 7a), as suggested in the task or a rectangle (Fig. 7b).

However, in cases like CE and BD in Fig. 6b, the resulting quadrilateral DEGH is an ordinary quadrilateral (Fig. 7c) and can even be non-convex (Fig. 7d). Quadrilaterals like those in Fig. 7b–d constitute general counterexamples to the conjecture that the quadrilateral DEGH is always a trapezoid. We refer to this conjecture as the 'trapezoid conjecture'.

Note that the construction of the counterexamples in Fig. 7c and d is not a trivial undertaking. While it is always possible to construct segments CE and BD such that a resulting quadrilateral DEGH is a trapezoid (Fig. 7a), the quadrilaterals like in Fig. 7c and d do not necessarily exist. This adds to the complexity of the construction of a counterexample that refutes the 'trapezoid conjecture'.

When the teachers encountered the isosceles triangle task, they first tried to draw their own examples of the problem situation in order to gain a better understanding of it. However, all the examples that they constructed were similar to Fig. 7a, and provided evidence in support of the trapezoid conjecture.

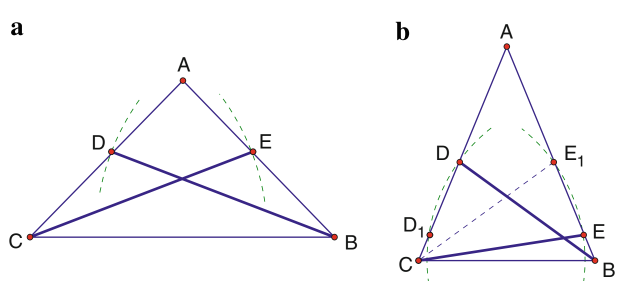


Fig. 6 **a** Two arcs of the same radius intersect each side of a triangle at a single point. **b** Two arcs of the same radius intersect each side of a triangle at two distinct points

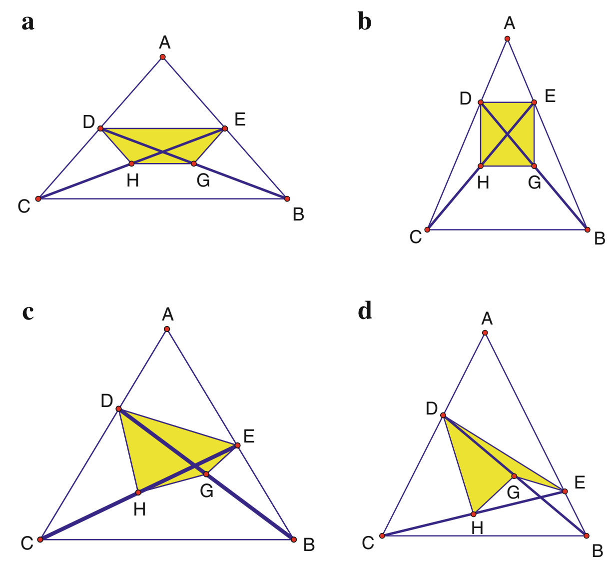


Fig. 7 **a** An example in which a quadrilateral DEGH is a trapezoid. **b** An example in which a quadrilateral DEGH is a rectangle. **c** An example in which DEGH is an ordinary quadrilateral. **d** An example in which DEGH is a non-convex quadrilateral

The teachers assumed that they would be able to provide valid justification for their assertion that if triangles ΔCBD and ΔBCE are congruent, then quadrilateral DEGH is indeed a trapezoid (e.g., if the segments BD and CE are medians). They now focused on trying to either prove or refute the claim that triangles ΔCBD and ΔBCE are congruent, completely overlooking the possibility that the quadrilateral could be a rectangle (note that according to the mathematical definition in their school curriculum, a rectangle is not a special case of a trapezoid). Their comments indicated that they were aware that triangles ΔCBD and ΔBCE may not be congruent, but the examples of trapezoids that they had constructed presented a strong visual support in favor of the trapezoid conjecture, which they considered a result of the congruence of these triangles. This resulted in a sense of uncertainty. In the next excerpt, Natalie referred to the trapezoid conjecture indicating her sense of uncertainty and reflected on her ability to resolve it:

> What bothers me is that I stopped here. I wasn't sure that I can prove that it's always true, and at the same time I wasn't sure that I can prove that it's false. Since we are talking about an isosceles triangle, it has to be so. It, sort of, seems true. But if a student would say such a thing to me, I wouldn't accept this as an

> argument. Not at all! […] We have to find this out in some way. I couldn't prove it, but I couldn't refute it either….

The teachers appeared stuck and did not know how to proceed. They asked the facilitator to show them a DG demonstration in order to gain additional insight about their conjecture. The facilitator attended to their request and unintentionally shared with them a DG demonstration that contains just a few supporting examples to the trapezoid conjecture, which did not provide the full range of possible outcomes.

It turned out that the DG demonstration reinforced the (invalid) trapezoid conjecture. The teachers were now completely convinced that this conjecture holds and were determined to prove that the sufficient condition with which they had come up earlier holds, i.e., that ΔCBD and ΔBCE are congruent. From here on, we can refer to their state as *strong confidence* in the conjecture, and as shown below, this confidence led them to 'prove' their assertion at any cost, even by flawed reasoning.

Debora stated that triangles ΔCBD and ΔBCE have two, respectively equal sides ($BD = CE$ and $BC = CB$), but she could not find a way to show that $\triangleleft DBC = \triangleleft ECB$.

After several unsuccessful proving attempts, Dafna came up with a flawed idea how to show that the triangles must be congruent. She accompanied her explanation by several drawings (Fig. 8).

The following excerpt contains Dafna's explanation, as she presented it to the teachers.

> There are only two possibilities: either triangles ΔCBD and ΔBCE are congruent or not. We have two pairs of respectively equal sides ($BC = CB$ and $BD = CE$), but the angle does not correspond to the longer side, but rather to the shortest one (Fig. 8a). So, if the triangles are congruent, the angles $\triangleleft ECB$ and $\triangleleft DBC$ are equal (since everything is equal). Now, suppose the triangles ΔCBD and ΔBCE are not congruent. If two general non-congruent triangles have two pairs of equal sides, they must have two angles that complete each other to 180° [like angles $\triangleleft MNK$ and $\triangleleft MLN$, in Fig. 8b]. If that would be our case, the sum of the angles $\triangleleft ECB$ and $\triangleleft DBC$ would be 180°. And this implies that the segments BD and CE in triangle ΔACB would be parallel, and such triangle can't exist—then ΔACB is not even a triangle. However, we know that a triangle ΔACB exists—since it is given. Hence, this alternative is impossible and triangles ΔCBD and ΔBCE must be congruent. Even if we don't have all conditions required by the congruence theorem, if we've ruled out the other alternative, then we can infer that the triangles are congruent.

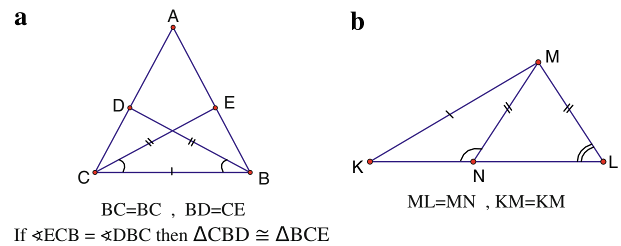


Fig. 8 Drawings that Dafna used to support her explanation

Dafna, as well as all other teachers, were so convinced that the triangles are necessarily congruent, to the extent that they did not check whether this had been correctly proven. In fact, Dafna incorrectly applied the case in Fig. 8b to Fig. 8a, by looking at the angles between the two pairs of corresponding sides (CE & CB and BD & BC) instead of the angles opposite the longer side ($\triangleleft CEB$ and $\triangleleft BDC$). The latter would not yield any contradiction. Moreover, it could have led to counterexamples such as in Fig. 7c and d. Interestingly, none of the teachers detected any flaws in her reasoning. On the contrary, they were all convinced and satisfied that a "proof" was finally found. It appears that the group was determined to wrap the case and sufficed by Dafna's arguments that had the flavor of a valid proof. The fact that Dafna's suggestions to the previous tasks were successful may have added to the trust the group had in her "proof". Yet, this fact strengthens this finding, since the flawed reasoning came from a capable and knowledgeable teacher.

The examples that confirmed the trapezoid conjecture, which teachers encountered through the DG demonstration, seemed to contribute to their strong confidence that the conjecture is true. This confidence, which was based on empirical visual examples, increased teachers' motivation to prove the conjecture at any cost.

4 Discussion

In this article, we presented findings based on a special type of task "Is this a coincidence?", which we used with high school students and with experienced mathematics teachers. Despite the differences in the settings and in the participants' background and orientation, we observed similar encounters in both groups. This type of task was found to have a potential for creating a need to prove or convince, either due to a sense of uncertainty regarding the mathematical phenomenon presented or an overconfidence in a false conjecture. The task fostered a need to prove that the phenomenon is not 'a coincidence' or to construct a counterexample refuting it. This need emerged spontaneously as a way to resolve an uncertainty, evoked by the task.

We attribute the emergence of uncertainty to the special design of the tasks. In particular, to the fact that we carefully selected mathematical phenomena that were not familiar to the participants and for each one we provided a specific example that supports an implicit conjecture about the generality of the phenomenon. In most cases, when the participants, both students and teachers, examined the confirming example, they experienced a sense of uncertainty whether the observed phenomenon is 'a coincidence' or not. The need to resolve this uncertainty appeared to be a trigger in creating a need for convincing and proving. As Fischbein (1987) pointed out the 'quest for certitude' is a fundamental and consistent tendency of the human mind.

Our findings concur with other studies that investigated the role of uncertainty in promoting a need to prove for students (Hadas, Hershkowitz & Schwarz, 2000) and for teachers (Zaslavsky, 2005; Buchbinder & Zaslavsky, 2008). The findings also show that the intensity of the uncertainty evoked by the task may vary. A strong sense of uncertainty motivated the construction of proofs, like in the case of the students Kerry and Ben (Case 1, the right triangle task) and in the case of the teachers (Case 4, the triangle areas task). Interestingly, even a rather moderate feeling of uncertainty resulted in a search for proof or disproof of a conjecture, as in the case of Alan and David (Case 2, the inscribed quadrilateral task) and in the case of the teachers (Case 5, the perpendicular rays task). An important common feature in the reaction of all the participants to this kind of uncertainty was the caution with which they treated the example in the task and their reluctance to jump to conclusions prior to exploring the problem.

A close look at the cases that led to disproof, as the teachers' solution of the perpendicular rays task (Case 4) and Alan and David's solution of the inscribed quadrilateral task (Case 2), indicates a common approach, which we term a 'negation' strategy. In order to check if a certain property is imperative, they tried to repeat the described construction trying to come up with an example that does not have the observed property. This strategy helped them gain understanding of the problem and its scope and limitations.

Interestingly, in several cases (as in Case 5), this approach led the participants to question the existence of the example that the hypothetical student obtained. The issue of existence of an example, particularly in geometry where examples are roughly sketched and assumptions are made regarding them, often with no warrants, it is important to consider this question (Zodik & Zaslavsky, 2008).

In some cases, instead of the intended uncertainty that the task aimed to create, either right away or in the course of dealing with the task, a strong sense of confidence in a phenomenon was evoked. We presented two such cases: Gila and Tami's solution to the inscribed quadrilateral task (Case 3) and the teachers' treatment of the trapezoid conjecture in the isosceles triangle task (Case 6). In both cases, a strong confidence in the generality of the observed phenomenon led to an incorrect assertion. Furthermore, in order to support such assertion, they manifested flawed reasoning and came up with an incorrect "proof". A close look at the participants' thinking processes in both cases (Cases 3 and 6) suggests that their flawed reasoning complies with one of the types of distortion of theorems that Movshovitz-Hadar, Inbar, and Zaslavsky (1986) found among students. More specifically, both students and teachers manifested a distortion of the antecedent, done by applying the claim within distorted conditions. In Case 3, Gila and Tami distorted the theorem that 'Any quadrilateral with perpendicular diagonals that bisect each other is a rhombus', by omitting the condition that the diagonals need to bisect each other. Dafna distorted the theorem that 'If two triangles have two sides that are respectively equal, then the measures of the angles opposite the longer side are either equal or of the sum of 180°', by applying the claim to the angles between the two pairs of corresponding sides. Note that the participants who exhibited this kind of flawed reasoning performed exceptionally well on the other tasks and appeared to be mathematically competent. This suggests that their strong conviction in their assertion and their determination to justify it at any cost led them to this logical carelessness. Even though such cases were rare, their analysis contributed to our understanding of the role of confirming examples in creating a drive to prove.

Our findings show that in addition to creating uncertainty regarding the generality of the conjecture implied by the tasks, the confirming examples that appeared in the tasks had several important roles. All examples presented a case for which the observed mathematical phenomenon holds. As such, they contributed to understanding of the task and the conjecture implied by it (Alcock, 2004). In addition, examples in the right triangle task and the perpendicular rays task allowed the participants to construct a correct proof (Case 1) or disproof (Case 5) of the conjecture. In both cases, careful examination of the drawing with respect to the described phenomenon contributed to construction of a correct solution.

The phenomenon presented in the inscribed quadrilateral task is in fact a 'coincidence', since it is only true for a square. However, the example was rather compelling and appeared to be misleading for some students, like Tami and Gali (Case 3), yet evoked suspicion and caution for other students, like Alan and David (Case 2).

These findings in the case of Tami and Gila (Case 3) concur with those of Behr and Harel (1995) who observed that students often resolved conflicts through application of erroneous procedures. The surprising observation for us was the occurrence of the same phenomenon with a group of experienced mathematics teachers. However, we emphasize that contrary to students, who worked with a paper and pencil version of the task, the teachers were exposed to examples constructed in a DG environment that have a different status than drawings (Mariotti, 2006). Compared to drawings, the convincing power of such examples is much more significant, to the extent that teachers constructed an incorrect argument supporting the false conjecture.

These findings suggest that lower level students and less knowledgeable mathematics teachers might be even more susceptible to making such mistakes. It follows that it is important to develop an awareness of the convincing power of confirming examples, critical thinking and control techniques.

It is a well known phenomenon that students tend to treat examples that confirm a mathematical claim as sufficient for proving it (e.g., Fischbein & Kedem, 1982; Harel & Sowder, 2007; Buchbinder & Zaslavsky, 2007). Using "Is this a coincidence?" task, we were able to detect additional students' (and teachers') reactions to confirming examples. These reactions varied from caution and reluctance to generalize, to confidence and conviction that the phenomenon manifested in the example is generally true. Nevertheless, adding to the findings documented in the literature, in all cases that we observed, confirming examples were considered as a starting point for exploration and proving and not as a substitute for proof. We attribute this effect to the special design of the task "Is this a coincidence?"

An important issue to address in creating uncertainty as a motivating force for proving is a need to maintain the appropriate level of uncertainty. Too strong uncertainty may result in frustration, inability to solve the problem and as a result in unwillingness to cope with it (Movshovitz-Hadar, 1993; Tsamir & Dreifus, 2005). Two features of the task design were introduced to prevent such scenario. First, the task is embedded in school curriculum context, and second, it is presented as an exploration performed by a student. This way, the participants, especially students, consider the solution within their reach and the uncertainty resolvable (in some way). We believe that these design principals contributed to participants' motivation to prove that the phenomenon observed in the task is a coincidence or is a general rule.

5 Concluding remarks

As we explained earlier, the findings regarding students were collected in the course of a larger study that deals with students' understanding of the roles of examples in determining the validity of mathematical statements (Buchbinder & Zaslavsky, 2009). In accordance with the goals of that study, the interviewer made no remarks on the correctness of students' solutions, nor did she encourage them to investigate further if they felt that they have reached a solution. It would be interesting to investigate similarities and differences in the processes that appear when this type of task is implemented in other settings, e.g., with larger groups of students, instead of pairs; or in a classroom environment with a teacher's mediation.

Our second note refers to the role of DG environments in the implementation of the task. Hadas, Hershkowitz & Schwarz (2000) indicate that role of uncertainty is particularly critical in DG environments, which in many cases "provide students with strong perceptual evidence that a certain property is true" (Mariotti, 2006, p. 193). Thus, once convinced by means of empirical evidence provided by DG, students often do not feel a need for additional validation in the form of deductive justification or proof. We used DG only with teachers in the form of demonstrations. Even in this limited form, it appeared to be compelling and influenced teachers' decision-making and their ways of dealing with uncertainty and doubt. It would be interesting to investigate the impact of working *with* DG environment in its full scope on the implementation of this type of task and the processes evoked by it for both teachers and students.

Our final remark concerns the advantages of "Is this a coincidence?" type of task for teachers. Teachers openly and spontaneously expressed their excitement and satisfaction with this type of task, both for their own learning and for their teaching. They explicitly said that they had gained a "valuable experience" from dealing with uncertainty in this kind of mathematically challenging task. The design and implementation of tasks for teachers that provide them with opportunities to engage in a stimulating mathematical activity, which is based on school mathematics, is a challenge for mathematics teacher educators (Zaslavsky, 2005, 2008). The task "Is this a coincidence?" meets many aspects of this challenge. Specifically, it elicited teachers' reflections on the role of examples in proving and on the potential of uncertainty in evoking the need for proving. This approach is referenced and described by Zaslavsky, Nickerson, Stylianides, Kidron, & Winicki (2011) as one of three main approached to facilitating the need for proof and proving.

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