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ORIGINAL ARTICLE

Examining the role of examples in proving processes through a cognitive lens: the case of triangular numbers

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Abstract In this paper, we analyze the role of examples in the proving process. The context chosen for this study was finding a general rule for triangular numbers. The aim of this paper is to show that examples are effective for the construction of a proof when they allow cognitive unity and structural continuity between argumentation and proof. Continuity in the structure is possible if the inductive argumentation is based on *process pattern generalization* (*PPG*), but this is not the case if a generalization is made on the results. Moreover, the *PPG* favors the development of generic examples that support cognitive unity and structural continuity between the argumentation and proof. The cognitive analysis presented in this paper is performed through Toulmin's model.

1 Background

It is hard to overestimate the role of examples in learning and teaching mathematics (Bills et al. 2006). Examples play an important role in concept formation (Vinner 1983; Tsamir, Tirosh & Levinson 2008), the illustration of new techniques and ideas (Watson & Mason 2005), the construction of generalizations (Alcock 2004), the clarification of meanings of new concepts (Dahlberg & Houseman 1997), and mathematical exploration. Goldenberg & Mason (2008) address examples as "a major means for 'making contact' with abstract ideas and a major means of mathematical communication whether 'with oneself', or with others" (p. 184). What makes examples so important is that they are the junction between the general and the particular. Every example becomes a carrier of an abstract idea when someone perceives it as a general instance, seeing beyond the particular.

Examples are often used by students in argumentation activities to construct and/or to justify a conjecture. Many researchers (e.g. Mason & Pimm 1984; Balacheff 1988; Harel 2001) draw attention to the way a *generic example* may point to a general argument, and highlight the reason why an assertion is true. A generic example is a single, particular object that represents the general case and through which a generality can be perceived. By ignoring the specifics of the example and by paying attention to its underlying structure, it might be possible to determine the general rule for generating other examples that contain the same features, as well as to identify the relationships between similar cases.

Watson & Mason (2005) suggest that exemplification can be seen as a "necessary but implicit part of inductive reasoning" (p. 189) especially in tasks that call for the identification of a general pattern for the nth term of a sequence. However, the research shows that such tasks present many challenges for the students who are required to move beyond examining the particular objects in order to construct a general conjecture and justify it (Lannin, Barker & Townsend 2006).

Through examining students' inductive reasoning, Harel (2001) observed two different types of generalizations

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which correspond to "two distinct ways of thinking": *result pattern generalization* and *process pattern generalization*. "In *process pattern generalization* (*PPG*) students focus on regularity in the process, whereas in *result pattern generalization* (*RPG*) on regularity in the results" (Harel 2001, p. 191). Harel (2008) exemplifies these concepts through students' solutions to the problem: "Prove that 1+3+5+…+(2n-1)=n^2 for all positive integers n". Some students focus only on the pattern of computed results: 1^2=1, 1+3=4=2^2, 1+3+5=9=3^2 in order to generalize that 1+3+5+…+(2n-1)=n^2. Other students approached the problem by writing the sequence of examples in a different way: 3^2+7=16, 4^2+9=25, 5^2+11=36. This representation enables the observation of the pattern: n^2+(2n+1) which leads to $(n+1)^2$ through PPG. Harel (2008) highlights the distinction between RPG and PPG by pointing out that "while in RPG one's conviction is based on regularity in the results—obtained, for instance, by substituting numbers—in PPG one's conviction is based on regularity in the process, thought it might be initiated by regularity in the results" (p. 118). Note that choosing a representation that will favor the PPG is not a trivial matter.

According to Harel (2008), only students who generalize through PPG propose arguments that resemble Mathematical Induction.

These results concur with the ones obtained by Pedemonte (2007) who observed that PPG appears to be required for the construction of mathematical inductive proof. In addition, the recent study by Martinez and Pedemonte (2011) suggests that similar results hold when the proof is deductive, at least in the case of calendar problems. These algebraic problems are intended to promote the study of regularities in the calendar and the construction of proof that justifies them. Given the nature of the problems, students are required to use numeric examples in inductive exploration in order to find a generalization. It was observed that only when students' inductive argumentation was grounded in PPG, they were able to construct a deductive proof. On the contrary, when inductive argumentation was grounded in RPG, students were not able to construct a proof.

In this paper, we try to further analyze students' argumentation processes involved in proving a conjecture that has been constructed through examining examples. We propose a cognitive analysis which attempts to enhance our understanding of why students sometimes are not able to use examples in effective ways. At the same time, we try to characterize arguments in which examples are used in fruitful ways. Our analysis is based on two theoretical constructs, which we discuss in the next section: *cognitive unity* and *structural continuity*.

2 Theoretical framework

2.1 *Cognitive unity* and *structural continuity* between argumentation and proof

During problem solving, argumentation is used both in conjecture production and in the construction of its proof. Studies that analyzed the relationship between these modes of argumentation (Boero, Garuti, & Mariotti 1996; Pedemonte 2007, 2008) highlight the continuity in the relationship between conjecturing and looking for a proof. This continuity is called *cognitive unity*. The hypothesis of *cognitive unity* states that in some cases an argumentation that led to the production of a conjecture can be used by the student in the construction of proof by organizing in a logical chain some of the previously produced arguments. Experimental research on cognitive unity (Boero et al. 1996; Garuti, Boero, Lemut & Mariotti 1996; Garuti, Boero, Lemut 1998; Mariotti 2001) showed that proof is more "accessible" to students if there is *cognitive unity* between argumentation and proof (e.g. when some words, drawings, theorems used in the proof have already been used in the argumentation supporting the conjecture).

However, further research in this field (Pedemonte 2007) showed that the analysis of *cognitive unity* does not cover all aspects of the relationship between argumentation used to construct a conjecture and its proof. Other kinds of continuity need to be taken into account. In particular, it seems important to consider the different *structure* that argumentation and proof may have. By *structure* we refer to a logical cognitive connection between statements (Pedemonte 2007) that corresponds to different types of reasoning: inductive, deductive or abductive. While a proof usually has a deductive structure, this is not necessarily the case for argumentation, which may also have abductive or inductive structure.

There is a *structural continuity* between argumentation and proof when they have the same structure (Pedemonte 2007).

Experimental research (Pedemonte 2007) shows that *structural continuity* does not support the construction of proof in the geometrical domain and might even present an obstacle for it. This might happen even when *cognitive unity* is present. The structure of argumentation is usually not deductive. Thus, in order to construct a deductive proof, it is often necessary for a student to convert the steps of his argumentation into deductive steps. The study showed that sometimes students are not able to construct a deductive geometrical proof because they are not able to transform abductive steps used in argumentation into deductive ones.

However, this result cannot be generalized in the algebraic domain where *structural continuity* is not necessarily an obstacle for students (Pedemonte 2008). Since algebraic proof is characterized by a strong deductive structure, abductive steps in the argumentation can be useful in linking the meaning of the letters used in the algebraic proof with numbers used in argumentation. Thus, unlike the geometrical case, abductive steps in argumentation can be helpful for the construction of proof if they favor *cognitive unity* between the argumentation and proof.

These different results show that cognitive analysis of argumentation and proof requires the consideration of two types of analysis: the one of *cognitive unity* and the other of *structural continuity*. Moreover, the mathematical domain in which argumentation and proof are performed has to be taken into account.

In this paper, we consider the role of examples in proving processes from a cognitive point of view. We employ these two types of analysis to cases in which argumentation involves examples. In particular, we explore whether *cognitive unity* between argumentation and proof can be realized in this case, and whether the pattern of generalization (RPG or PPG) in inductive argumentation affects the construction of a deductive proof.

2.2 Constructive argumentation and structurant argumentation

The comparison between argumentation supporting a conjecture and its proof is based on the hypothesis that proof can be considered as a particular argumentation in mathematics (Pedemonte 2007). According to contemporary linguistic theories (Plantin 1990; Toulmin 1993), argumentation as proof is a set of rational justifications expressed as inferences. These inferences can be analyzed and compared with Toulmin's model which can be used to detect and analyze argumentation supporting a conjecture and proof.

In analyzing students' protocols, we consider the whole argumentation process performed by students to solve the problem: the argumentation connected to the conjecture and the proof. By this we refer to the particular argument that a student perceives as a "proof". However, in order to analyze the relationship between argumentation and proof, it is important to distinguish between them in students' protocols.

The argumentation can be related to the conjecture in two ways: we term the argumentation that contributes to the construction of a conjecture a *constructive argumentation*. Thus, it precedes the formulation of the conjecture. The argumentation that justifies a conjecture, previously constructed as a "fact", and that comes afterwards is termed a *structurant argumentation* (Pedemonte 2007).

The constructive and structurant argumentation can both be present in the resolution process, e.g. when a constructive argumentation involves inductive reasoning based on RPG. While this argumentation allows a person to construct a conjecture, it is not sufficient for justifying it. Thus, a construction of proof also requires structurant argumentation. Note that the structurant argumentation and the proof can be either distinct from each other or coincide. Moreover, constructive and structurant argumentation could also be coincident when argumentation allows the user both to construct a conjecture and to justify it.

Referring to the previously discussed concepts, we say that *cognitive unity* is realized when there is continuity in the content between constructive and/or structurant argumentation and proof. *Structural continuity* is realized when constructive and/or structural argumentation and proof have the same logical structure (abduction, induction or deduction).

Examples may be present both in constructive and in structurant argumentation, but they play two different roles. In constructive argumentation, they can be useful in the construction of a conjecture, while in structurant argumentation they can be used to justify a previously constructed conjecture. In the last case, students can perceive the examples as sufficient for convincing themselves or others of the correctness of the conjecture. On the contrary, generic examples can be used in the structurant argumentation to identify general constructs that can be used for the construction of a proof. There might also be additional ways in which examples can be used in both constructive and structurant argumentation. In order to analyze the role of examples in argumentation and proof, we use Toulmin's model which is presented in the next section.

2.3 Toulmin's model for analyzing the structure of argumentation

Through Toulmin's model it is possible to represent the whole structure of argumentation in which examples are used. In the educational literature, Toulmin's model has been already used for detecting and analyzing how learning progresses in the classroom (Inglis et al. 2007; Krummheuer 1995; Yackel 2001; Yackel & Rasmussen 2002), how a classroom environment that promotes argumentation can be created (Wood 1999), and how structures of argumentation in classroom proving processes can be revealed (Knipping 2008). It is also used to compare student's argumentations and their proofs from a cognitive point of view (Pedemonte 2005, 2007, 2008).

In Toulmin's model, an argument is comprised of three elements (Toulmin 1958, 1993):

C (*claim*): the statement of the speaker,
D (*data*): data justifying the claim C,
W (*warrant*): the inference rule, which allows data to be connected to the claim.

In any argument, the first step is expressed by a standpoint (an assertion or an opinion). In Toulmin's terminology, this standpoint is called the *claim*. The second step

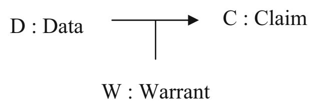


Fig. 1 Toulmin's basic model

consists of the production of *data* supporting the claim. The *warrant* provides the justification for using the data conceived as a support for the data–claim relationship. The warrant, which can be expressed by a principle, or a rule, acts as a bridge between the data and the claim.

The basic structure of an argument is presented in Fig. 1.

Three auxiliary elements may be necessary to fully describe an argument: a backing, a qualifier, and a rebuttal (Toulmin 1958, 1993). A *backing* is used to support and strengthen the warrant. The *qualifier* expresses the degree of confidence in the claim, and the *rebuttal* is used to state the conditions or exceptions to the general claim.

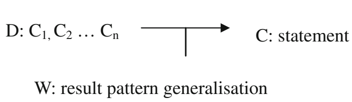
These additional components might be essential for describing the structure of argumentation as was recently shown by Inglis, Mejia-Ramos and Simpson (2007) who conducted an empirical study with high achieving students. In this study, students constantly used qualifiers and rebuttals while solving problems in number theory.

In the present study, we focus on the role of examples in argumentation. It is reasonable to assume that examples can play an important role in the qualifier and in the rebuttal. Examples can influence the qualifier if they are used to support the conjecture that has been previously constructed. Counterexamples can weaken the argument, playing the role of the rebuttal, if they are used to specify the cases for which the argument does not hold. However, the data suggest that in the present study the additional components of Toulmin's model did not play the significant role that might be expected. This may be attributed to the specific nature of the problem: finding a general rule for triangular numbers.

2.3.1 Inductive argumentation in Toulmin's model

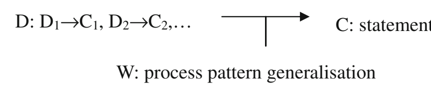
As shown in previous research (Pedemonte 2007), Toulmin's model is an effective tool to represent inductive argumentation and proof. Here, the example/s can be considered as data while generalization is the warrant. Different types of generalizations are possible. Consequently, different uses of examples can be described.

As highlighted before, Harel (2001) distinguishes between two different types of generalizations present in an inductive process: *RPG* and *PPG*. In Toulmin's model, *RPG* can be represented as follows:



C_1, C_2 …C_n are the claims produced in the previous steps. In our case, C_1, C_2 …C_n represent claims derived by the use of examples.

Since *PPG* focuses on regularity in the process, it can be represented, in Toulmin's model, as follows:



$D_1 \rightarrow C_1$, $D_2 \rightarrow C_2$… are the previous arguments constructed based on the use of examples.

3 The study

As mentioned before, we can distinguish between two different ways in which examples can be used in problem solving:

- To construct a conjecture, as a part of a constructive argumentation.
- To support a conjecture, as a constituent of structurant argumentation.

It is probable that examples that appear in a structurant argumentation are closer to a proof because they are used by the arguer to find justifications for the conjecture, which could mirror theoretical ones. On the contrary constructive argumentation has an explorative role.

We suggest that examples (mobilized both in constructive and structurant argumentation) are probably effective for the construction of a proof if they favor *cognitive unity*. *Cognitive unity* can be realized if examples used to construct and/or justify a conjecture can be generalized for the construction of a proof.

However, it is important to take into account the *structural continuity* between argumentation and proof. The logical concatenation in which examples are considered in argumentation is probably essential for establishing continuity in the structure between argumentation and proof. Consider the following hypothetical scenario. In the resolution process of an open problem, a constructive argumentation is based on RPG, but after a conjecture is formulated a solver is able to attend to an underlying structure of the series of examples and to justify the conjecture through structurant argumentation based on PPG. Following that move, a solver constructs a proof by Mathematical Induction. In such a scenario, there is a structural gap between constructive argumentation and proof. Nevertheless, due to the presence of structurant argumentation between the constructive argumentation and proof, there is a *structural continuity* between argumentation and proof in the solution process. Such a scenario suggests that the structurant argumentation has probably a fundamental role to play in supporting *structural continuity* between argumentation and proof.

As Harel (2008) pointed out, the base step of Mathematical Induction (show P(1)) is given by the *RPG*, while the inductive step (show $P(n) \Rightarrow P(n+1)$ for every positive integer n) is based on *PPG*. Thus, it seems that in order to construct a proof, examples involved in a structurant argumentation should be connected through an induction based on a *PPG*.

Our study was designed to meet the following goals:

- To examine the role of examples in resolution of open problems, with the specific focus on their role in constructing and proving a conjecture.
- To characterize arguments in which examples are successfully or unsuccessfully used in the construction of proof.

4 Methodology

The context chosen to study the role of examples in argumentation was a problem of finding a general rule for triangular numbers. A triangular number is the number of dots in an equilateral triangle evenly filled with dots (see Fig. 2).

For example, three dots can be arranged in a triangle; thus, 3 is a triangular number. The *n*th triangular number is the number of dots in a triangle with *n* dots on a side.

The problem of finding a general rule for the *n*th triangular number was chosen because it calls for spontaneous use of examples in the construction of a conjecture. In order to detect the general pattern of triangular numbers, it is necessary to construct several such numbers and to observe both the structure of each individual number with respect to its place value and the connection between consecutive numbers as well.

This problem was proposed in Italy to 12 students in the last year of upper secondary school (17–18 years old). Although it is not a part of the regular school curriculum at this level, the students' mathematical background is sufficient for solving the problem. The teacher (the first author of this paper) introduced the concept of triangular numbers by drawing on the board the first five triangular numbers. Then, she proposed the following problem individually to some students (outside the lessons hours):

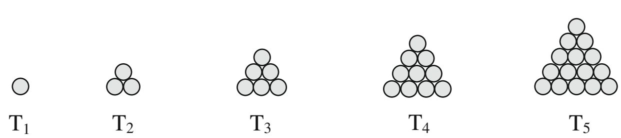


Fig. 2 First five triangular numbers

"Consider a triangular number. Can you find the number of dots in a triangle with n dots on a side?"

The problem was proposed on a sheet of paper, and did not include any drawings. The teacher explicitly required that students construct a proof to justify their answers.

Data were collected through individual interviews in which the students were asked to solve the problem aloud. The teacher did not intervene in the solution process. The students were notified that they would not be evaluated according to their success in solving the problem and any mistakes they made would not be taken into account.

The students' argumentation and their proofs were analyzed by means of Toulmin's model. As we shall show in the following section, it was interesting to observe the different ways in which similar examples were used by different students. In particular, in some cases, students were able to construct a proof while in other cases, using similar examples, a proof was not constructed.

5 Findings

In this section, we present a detailed analysis of four solutions to the triangular numbers problem. These cases illustrate different ways in which examples are used in the resolution process. In the first two cases, students constructed conjectures but could not prove them. In the other two cases, students constructed conjectures and proofs to justify them.

The text was translated from Italian into English. The students' utterances and the descriptions of their actions (in italics) appear in the left hand column, while comments and analysis are reported in the right hand column. The subscripts identify each argumentative step.

5.1 Case 1: Generalization is based on the results. No proof is constructed

Matteo considers some numerical examples in the constructive argumentation. He produces a conjecture through *RPG*.

Transcript	***Analysis***
Matteo writes: n=1 R=1 n=2 R=2+1 n=3 R=3+2+1 Each time we add the last number. For a triangle with n dots on a side R_n is n+(n-1)+... +1 *After a few minutes he asked the teacher* Am I obliged to prove this result? Because I don't know how to do it... *The teacher says that it is important to prove the obtained result. However, Matteo does not produce a proof.*	*Matteo constructs the conjecture based on a generalization of the results: he finds a general law by examining numeric examples.* *In Toulmin's model we can represent the constructive argumentation as follows:* D: C_1, C_2, C_3, → C: R_n =n+(n-1)+... +1 W: *result pattern generalization* *Where C_1, C_2, C_3, are the statements produced by the arguments constructed for each example:* D_1: if n=1 → C_1: R=1 D_2: if n=2 → C_2: R=2+1 D_3: if n=3 → C_3: R=3+2+1

Matteo does not construct a structurant argumentation to justify the conjecture. He is also not able to construct a proof. Matteo's use of the examples is not effective for the construction of a proof.

5.2 Case 2: Generalization is based on the process. No proof is constructed

At the beginning, Lara considers some numerical examples in the constructive argumentation. The conjecture is produced by focusing on the relationship among the results given by numerical examples. Lara connects each example with the previous one. Thus, an inductive argumentation based on PPG is constructed.

Transcript	***Analysis***
If n is 1 then the result is 1...If n=2 the result is 2+1 that is 3... if n=3 then we have 3+2+1 that is 6, if n is 4 then 4+3+2+1 is 10...Each time we add the last number... n=1 R=1 n=2 R=2+1=3=1+2 n=3 R=3+2+1=6=3+3 n=4 R=4+3+2+1=10=6+4 In general for a triangle with *n* dots on a side we have $R_n=R_{n-1}+n$ Now I have to prove the rule butI do not know how. However I'm sure it is correct.	*Lara constructs the conjecture as a generalization on the process. She considers the results of the four consecutive examples and finds a general law connecting each argument with the previous one.* *In Toulmin's model we can represent the constructive argumentation as follows:* Q: I'm sure it is correct D: $C_1, C_1 \rightarrow C_2, C_3 \rightarrow C_4$ → C: $R=R_{n-1} + n$ W: process pattern generalization *Where $C_1, C_1 \rightarrow C_2, C_2 \rightarrow C_3$ are the arguments in which each example is connected to the previous one.* *Each argument can be represented in Toulmin's model. For instance, representation for $C_1 \rightarrow C_2$ and $C_2 \rightarrow C_3$ are:* $D_1=C_1$: for n=1, R=1 → C_2: for n=2 R=2+1=3=1+2 W: computational rules; Rule: we add 2 to R_1 $D_2=C_2$: for n=2, R=3 → C_3: for n=3 R=3+2+1=6=3+3 W: computational rules; Rule: we add 3 to R_2 *The qualifier Q of the argument $D \rightarrow C$ is explicit. Lara says that even if she is not able to produce a proof she is sure that the formula is correct. However, the strength of the conviction does not affect the production of a proof.*

In the constructive argumentation, Lara considers a series of examples and constructs the conjecture, which is a recursive rule: $R_n=R_{n-1}+n$. Lara's argumentation is inductive, based on a PPG. She focuses on the way in which the consecutive examples are connected. Nevertheless, she does not produce a proof, probably because she is not familiar with proof by Mathematical Induction. In order to realize *structural continuity* between the type of argumentation which is present in Lara's case and a possible proof, a proof by Mathematical Induction should be constructed. In contrast, in the next case, *structural continuity* is realized.

5.3 Case 3: Generalization is based on the process. The proof is constructed

Sara considers the same examples as Lara in the previous case, but in a different way. Sara constructs a conjecture through constructive argumentation based on RPG, and justifies it through a structurant argumentation in which the focus is on the process. Consider Sara's constructive argumentation.

Transcript	***Analysis***
Sara writes n=1 1 n=2 2+1 n=3 3+2+1 n=4 4+3+2+1 n=5 5+4+3+2+1 n=6 6+5+4+3+2+1 n=7 7+6+5+4+3+2+1 *Then she observes what she has written and says*: If I consider a triangle with *n* dots on a side the result is: n+(n-1)+ ....+1	*Sara probably constructs the conjecture as a generalization of the results. In this way she finds a general law.* *In Toulmin's model we can represent her constructive argumentation as follows:* D: $C_1,C_2,\ldots C_7$ → C: for a triangle with n dots the result is n+(n-1)+ ....+1 W: *result pattern generalization* *Where $C_1, C_2, \ldots C_7$ are statements produced by arguments constructed for each example.* *The representation of these arguments in Toulmin's model is completely analogous to those represented in the case 1:* D_1: if n=1 →C_1: R=1 D_2: if n=2 →C_2: R=2+1 …. D_7: if n=7→C_7: R=7+6+5+4+3+2+1

In constructive argumentation, examples are used to construct a first conjecture that is "for a triangle with *n* dots the result T_n is: n+(n-1)+…+1". In this case, generalization seems to be made on the results. Nevertheless, in the structurant argumentation, as we show in the following, a generalization is made on the process and examples are used to reorganize the previously constructed expression.

Two examples are considered: $n = 7$ and $n = 6$. These examples assume the role of generic examples for Sara.

Transcript	*Analysis*
Perhaps we can write it in a better way... If we add the first number of each case to the last one...and the second number to the number before the last one...we obtain the same result.	*Now Sara considers examples in a different way. She tries to justify her conjecture by writing it in a different way. Here the focus is on the two specific examples n=7 and n=6.*
Lara observes the last example (n=7) but she also refers to the other examples (n=6, n=5).	
The sum of the first number (7) and the last one (1) is the same as the sum of this one (6) and (2) that is 8. And also 5 and 3, but there is a 4 that remains. If n=6 the sum is always 7... there is no number left in the middle...	D_1: n=7 → C_1: the number of dots is (7+1)·3+4 W: computational rules
She writes: n=7 7+6+5+4+3+2+1 n=6 6+5+4+3+2+1	D_2: n=6 → C_2: the number of dots is (6+1)·3 W: computational rules
If n is an even number I can write the sum of the n numbers from 1 to n as n/2 times (n+1). But if n is an odd number, the sum of the n numbers is: (n-1)/2 times (n+1) plus (n+1)/2 which is the number in the middle	*At this point Sara generalises* D_3: $D_2 \rightarrow C_2$ → C_3: if n is an even number the sum of the n numbers is (n+1) times n/2 W: *process pattern generalization*
She writes: $(n+1)\cdot\frac{n-1}{2}+\frac{n+1}{2}=$ $\frac{n+1}{2}\cdot(n-1+1)=\frac{n+1}{2}\cdot n$ which is the same!	D_4: $D_1 \rightarrow C_1$ → C_4: if n is an odd number the sum of the n numbers is (n+1) times (n-1)/2+(n+1)/2 W: *process pattern generalization*
Then the sum of the n numbers from 1 to n is always: $\frac{n\cdot(n+1)}{2}$. *Later Sara constructs a proof*	

Sara constructs a new conjecture: "The result T_n is: $\frac{n\cdot(n+1)}{2}$". Here, examples ($n = 7$ and $n = 6$) are used to justify it. In fact, Sara considers n separately as even or odd number while generalizing the process observed in the two examples. The generic examples given by the cases $n = 6$ and $n = 7$ are extremely significant to Sara: they allow her to make a generalization about the process and consequently to construct a proof. Note that the arguments $D_1 \rightarrow C_1$ and $D_2 \rightarrow C_2$ have a deductive structure. This structure is later used to prove the generic formula through algebraic rules. Following is the proof:

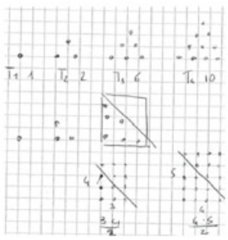
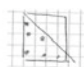
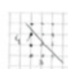
Transcript	*Analysis*
Sara writes: If n even number $\rightarrow T(n)= \left.\begin{matrix}1+n \ + \\ 2+n-1 \ + \\ 3+n-2 \ + \\ \dots\end{matrix}\right\}$ n/2 times T(n)=(n+1)·n/2	D_5: n is an even number → C_5 :T(n)= (n+1)·n/2 W: algebraic rules
If n odd number $\rightarrow T(n)= \left.\begin{matrix}1+n \ + \\ 2+n-1 \ + \\ 3+n-2 \ + \\ \dots\end{matrix}\right\}$ (n-1)/2 times +(n+1)/2	D_6: n is an odd number → C_6: T(n)= (n+1)·n/2 W: algebraic rules
Then $T(n) = \frac{n\cdot(n+1)}{2}$ for each case.	D_7: $D_5 \rightarrow C_5$ $D_6 \rightarrow C_6$ → C_7: T(n)= (n+1)·n/2 for each n W: all n numbers are considered (even and odd)

In Sara's solution, there is *cognitive unity* between argumentation and proof and there is also continuity in structure between structurant argumentation and proof. In fact, the proof is deductive in the same way the arguments $D_1 \rightarrow C_1$ and $D_2 \rightarrow C_2$ are. These arguments were derived from inductive argumentation based on PPG and on the examination of generic examples: $n = 6$ and $n = 7$. These arguments are: "If n is an even number then T(n) can be written as an even number of addends which can then be added in a specific way (the first with the last and so on) giving the same results. If n is an odd number, T(n) can be written as an odd number of addends. These addends can then be added as in the previous case, but at the end of the process the middle number has to be added to the final sum".

The case of Sara illustrates that *structural continuity* between structurant argumentation and proof has an important role for the construction of a proof. This *structural continuity* is possible because structurant argumentation which is in part deductive for generic examples, and in part inductive—based on PPG, supports the construction of deductive proof (at least partially). Moreover, it seems that there is a direct connection between an inductive argumentation based on PPG and generic examples. Sara's inductive argumentation contributed to the emergence of generic examples, which were crucial for the justification of Sara's conjecture.

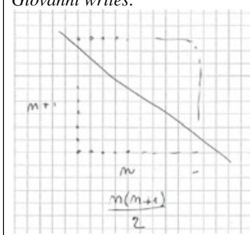
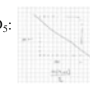
5.4 Case 4: Generalization is based on the process and involves visual examples. The proof is constructed

Giovanni considers visual examples: he remembers the figures constructed to define triangular numbers and rearranges the dots to construct right-angled triangles. Here, we look at his work in the following table.

Transcript	***Analysis***
Giovanni writes: 	D_1:  → C_1:  W: change of representation *Giovanni changes the representation transforming the arrangement of the dots in the triangle into a right-angled triangle of dots. There is a moment in which one of these triangles (the triangle T_3) becomes a generic example for him.*
When he considers the case T_3 he says I can construct a rectangle... it is a double triangle ....wait I have to make the triangle double because then I can divide it by 2.	D_2:  → C_2: Two copies of T_3 form a rectangle, which divided by 2 gives back T_3. W: visual implicit rule *The warrant of this argument is not explicit. Giovanni is able to visualise that a rectangle is made of two copies of T_3 and establishes a connection between it and T_3.* D_3  → C_3: The rectangle's width is 3 and its height is 4 W: area calculation for a rectangle *Giovanni counts the dots in the rectangle constructed of two triangles T_3.* *For T_3 Giovanni finds a rule:* D_4: C_3 → C_4: $T_3 = \frac{3 \cdot 4}{2}$ W: area calculation *After this he writes another argument for T_4, which is also a generic example for him. And then from this he can finally generalise.*

Giovanni's argumentation is based on drawings which represent the configuration of triangular numbers. He starts by making a change in the form of the visual representation of the triangular numbers. Note that he considers each triangular number individually, and looks at the structure of each one. He does not focus on the ways in which the examples are connected (as opposed to the approach used, for example, by Lara).

One of the examples that Giovanni considers (T_3) becomes a generic example for him and allows Giovanni to recognize its underlying structure. The case T_4 is used to confirm and reinforce his initial assertion. Both cases T_3 and T_4 are considered as generic examples and support the construction of a general conjecture. This argumentation is both constructive and structurant because it allows him not only to construct a conjecture but also to justify it.

Transcript	***Analysis***
Giovanni writes: 	*Giovanni constructs a visual proof.* D_5:  → C_5: $T_n = \frac{(n+1) \cdot n}{2}$ W: area calculation *The previous argument is deductive but Giovanni generalises the process that he discovers in the generic examples. The drawing of T_n is the result of a generalization of the pattern present in the drawings.* *Note that there are two implicit inductive arguments, where Giovanni generalizes the case n, based on PPG. They are:* D: $D_2 \rightarrow C_2$ → C: Two copies of T_n form a rectangle, which divided by 2 gives back T_n W: *process pattern generalization* D: $D_3 \rightarrow C_3$ → C: The n^{th} rectangle has sides: n and n+1 W: *process pattern generalization*

In this case, the process used to construct the solution is generalized: Giovanni constructs the proof considering a rectangle with sides: n and $(n + 1)$. The visual proof is constructed in continuity with Giovanni's argumentation not only because similar drawings are used (*cognitive unity*) but also because there is *structural continuity* between argumentation and proof. In each step, Giovanni considers the base and the height of a rectangle. First, for two generic examples and later, for the general case of T_n. In the later case, he applies the area formula for the triangle which is a half of a rectangle. The structure of this argumentation is deductive and it is used both in the generic examples and in the proof.

In a way similar to Sara's case, the generic example plays a crucial role in Giovanni's solution because it connects visual representation with algebraic rule.

6 Summary and discussion

Our analysis of the students' solutions to the triangular number problem focused on the role of examples in constructing and proving conjecture. It was aimed at characterizing arguments where students used examples productively and non-productively. We summarize our findings in Table 1.

Both Matteo and Lara used numeric examples in constructive argumentation. In contrast to Matteo, Lara's

Table 1 Summary of students' solution processes

Student	Type(s) of argumentation	Type of pattern generalization in inductive argumentation	Generic example	Type of argument for generic example	Type of proof	Rule constructed	Type of continuity
Matteo	Constructive	RPG	–	–		$R_n=n+(n-1)+\ldots+1$	–
Lara	Constructive	PPG	–	–		$R_n=R_{n-1}+n$	–
Sara	Constructive	RPG	–	Deductive	Deductive	$T(n)=\frac{(n+1)\cdot n}{2}$	CU
	Structurant	PPG	✓				SC
Giovanni	Constructive/ structurant	PPG	✓ (visual generic)	Deductive	Deductive	$T(n)=\frac{(n+1)\cdot n}{2}$	CU SC

CU cognitive unity, *SC* structural continuity

attention was focused on the connections between consecutive triangular numbers, that is to say, on the *process* of constructing a sequence of triangular numbers. Based on PPG, Lara was able to construct a recursive rule for triangular numbers, but proving her conjecture required the knowledge of proof by Mathematical Induction, which Lara did not have.

Sara and Giovanni were able to use examples successfully both for the construction of a conjecture and for proving it. Sara constructed her conjecture based on RPG of numeric examples in constructive argumentation and later justified it using PPG and generic examples in structurant argumentation. Giovanni, on the other hand, used visual generic examples and based his argumentation on PPG. Despite the differences in Sara's and Giovanni's argumentation processes and their use of examples, there are several common features to their solutions. Both Sara and Giovanni relied on PPG in the structurant argumentation, both used generic examples and they were both able to construct a deductive proof for their conjecture.

When a student uses examples in the constructive argumentation, the aim is to find a pattern that can be generalized in order to construct a conjecture. As our results show, a generalization of the results, without understanding the connection between the examples or the underlying structure of the example itself, can be sufficient for the construction of a conjecture, but it is usually not enough for the construction of proof. This is evident in Matteo's case. However, even when a conjecture is constructed through a PPG, it does not necessarily lead to a construction of a proof, as Lara's case shows. PPG allows *structural continuity* between inductive argumentation and proof by Mathematical Induction. Unfortunately, such continuity cannot be realized if a student is not familiar with this type of proof.

These findings point to the importance of the way in which examples are used in structurant argumentation. In our study, we have observed that proof was constructed by students only in cases in which *structural continuity* between argumentation and proof was possible (cases of Sara and Giovanni).

Sara and Giovanni both constructed a proof and in their cases there was *cognitive unity* and *structural continuity* between the argumentation and the proof. Sara and Giovanni, each in their own way, used examples effectively in their argumentations. These argumentations were based on a combination of PPG and the use of generic examples. As a matter of fact, the analysis suggests that it was the use of PPG that allowed the emergence of the generic examples, which in turn supported both *cognitive unity* and *structural continuity*.

The importance of a generic example stems from its ability to connect between arithmetic and algebraic domains. This connection supports *cognitive unity* because an example used in argumentation can be generalized through algebraic representation in the proof. This is because the rules used for examples, and in particular for generic examples, can be generalized in the algebraic domain.

In addition, a generic example supports *structural continuity* because usually argumentation used when students reason about a generic example has a deductive structure. This deductive structure is present in structurant argumentation and/or in the proof, as can be evident from the cases of Sara and Giovanni.

7 Conclusions

In this paper, we presented an analysis of four students' solutions to the triangular number problem. This type of problem requires a solver to identify the pattern, to generalize it by constructing a conjecture and then to justify it. Research literature and policy documents advocate the use of such activities to promote students' algebraic reasoning (Healy & Hoyles 1999; Lannin et al. 2006; Watson & Mason 2005; NCTM 2000). At the same time, the research shows that students encounter various difficulties in generalizing visual patterns. The results of our study concur with those of other researchers and contribute to them by highlighting the role of examples in the resolution process and characterizing the types of argumentation that favor the construction of conjecture (a generalization rule) and its proof.

In our study, students used examples both for the construction of conjecture and for its proof. The concepts of *RPG* and *PPG* proposed by Harel (2001) proved to be very useful in our analysis. The findings show that students were able to construct a conjecture using both types of generalization. However, only students who attended to the regularity in the process (PPG) of example generation or to the example's underlying structure were able to create conjectures that could later be justified by proof. This includes also a recursive rule, which can potentially be proved using Mathematical Induction.

Our analysis reveals that examples were effective for the construction of a proof only when they allowed *cognitive unity* and *structural continuity* between argumentation and proof. In addition, the type of generalization (RPG or PPG) that was used in structurant argumentation also affects the construction of the proof.

Structurant argumentation which has an inductive structure and is based on RPG is unlikely to lead to the construction of a proof. On the contrary, structurant argumentation, which is based on generic examples, and thus has a deductive structure, allows the construction of deductive proof (as in the cases of Sara and Giovanni). In these cases, generic examples served as a bridge between argumentation and proof by allowing both *cognitive unity* and *structural continuity* to be realized.

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